Stability Change of Periodic Orbits and the Phase Space Structure of Buckled Beam and Hénon-Heiles System

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Abstract. *This paper concerns the stability change of periodic orbits and the phase space structure of the system with Hamiltonian* $H = \frac{1}{2}(p_1^2+p_2^2)+\frac{1}{2}(-q_1^2+\omega_1^2q_2^2)+\frac{1}{4}q_1^4+\frac{j_2^2}{2}q_1^2q_2^2$*, particularly for case* $j_2 = 2$ *and* 6*. We study numerically this problem by two-dimensional Poincaré maps. Moreover, we analyse the results as the energy increases and calculate the Lyapunov exponents. We do a comparison to Hénon-Heiles system with Hamiltonian of the form* $H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2) + q_1^2q_2 - \frac{1}{3}q_2^3$ *and investigate the difference.*

Keywords: buckled beam, Hénon-Heiles, stability of periodic point, Lyapunov exponent

1 Introduction

Let us consider a special class of Hamiltonian system whose Hamiltonian has the form

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2) \tag{1}$$

with

$$V(q_1, q_2) = \frac{1}{2}(-q_1^2 + \omega_1^2q_2^2) + \frac{1}{N+1}q_1^{N+1} + \frac{j_2^2}{2}q_1^{N-1}q_2^2 + +\mathcal{O}(q_2^3).$$

Here $N \geq 3$ and $j_2 > 1$ are integers, and $\omega_1 = j_2\sqrt{(j_2^2\pi^2 - \Gamma)/(\Gamma - \pi^2)}$ with a constant Γ satisfying $\pi^2 < \Gamma < 4\pi^2$. This type of Hamiltonian is introduced by Yagasaki [5] in relation to a model equation (integro, partial differential equation) describing the undamped buckled beam. Actually, when $N = 3$ and $\mathcal{O}(q_2^3) = (j_2^4/4)q_2^4$, it corresponds to the two modes reduction of the model equation, where the constants j_2 and Γ denote respectively the mode of the Fourier expansion and the compressive force of the buckled beam [5]. The nonintegrability of the corresponding systems is studied in [5] using the idea of bifurcation of homoclinic orbits near a saddle-center equilibrium at the origin.

In this paper, we consider the case with $N = 3$ and without remainder term $\mathcal{O}(q_2^3)$. Namely, we study numerically Hamiltonian systems whose Hamiltonians are of the form (1) with

$$V(q_1, q_2) = \frac{1}{2}(-q_1^2 + \omega_1^2q_2^2) + \frac{1}{4}q_1^4 + \frac{j_2^2}{2}q_1^2q_2^2. \tag{2}$$

These systems are independent of the original buckled beam, but we will just name it buckled beam for the word efficiency reason. We will focus on two special cases with $j_2 = 2, 6$ and discuss about their orbit structures for large energy values as well as small ones.

The Hamiltonian system with Hamiltonian (1) and (2) is given by

$$\dot{q}_k = p_k \quad \dot{p}_k = -V_{q_k} \quad (k = 1, 2), \tag{3}$$

where the dot represents the differentiation with respect to time variable t. This is a two-degree of freedom system and has four-dimensional phase space. It is difficult to solve analytically in general. However, qualitative behavior of its solutions can be analysed by using two-dimensional Poincaré map. A fixed point or more generally periodic point of this map corresponds to a periodic orbit of the system (3). Also, an invariant closed curve corresponds to an invariant torus carrying quasi-periodic orbits. It was to the so called Hénon-Heiles system (4) that the numerical analysis based on this Poincaré map was first successfully applied. It was introduced as a model equation of motion of stars in galaxy and its Hamiltonian is given as

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2) + q_1^2 q_2 - \frac{1}{3}q_2^3. \tag{4}$$

The potential of this system is a cubic polynomial in contrast to the above case with fourth degree polynomial potential.

Among quantitative indicators of chaotic motions, the most basic one is the maximal Lyapunov exponent [1]. It measures the rate of divergence (deviation) of nearby orbits from a given one as $t \to \infty$. There is a large amount of literature for the numerical analysis concerning Lyapunov exponents. We refer to [4] for recent study of Lyapunov exponents for the Hénon-Heiles system.

The paper is organized as follows: In Section 2, we define the Poincaré maps for both buckled beam and Hénon-Heiles Hamiltonians. For Hénon-Heiles system, it is standard to take the plane $q_1 = 0$ as cross section. In this paper, however, we take $q_2 = 0$ as the cross section so that the origin $(q_1, p_1) = (0, 0)$ becomes a fixed point for the Poincaré map with any energy value. This holds for both buckled beam and Hénon-Heiles Hamiltonians. This fixed point corresponds to periodic orbits for original systems, and a complex analytic method for proving nonintegrablity of concrete Hamiltonian systems is based on the use of such periodic orbits (cf., [3]). In Section 3, we introduce the system of variational equations to analyse stability of the periodic orbits. We also introduce Lyapunov exponents in Section 4 to analyse their behavior as energy gets large. In Section 5, we show the results and discussions for the study explained in Section 3 and Section 4, applied to buckled beam and Hénon-Heiles system. Finally we present the conclusion in Section 6.

2 Poincaré maps

Let us consider the Hamiltonian system with Hamiltonian (1) with (2) or (4), and let Σ be the 3-dimensional surface defined by $q_2 = 0$ and Σ_E the intersection of Σ with the energy surface $H = E(= \text{const})$. Then the Poincaré map is defined by following the solution through a point P on Σ_E to next intersection Q with Σ_E. We denote by φ_E this Poincaré map. The relation $H = E$ can be used to eliminate p_2 so that the surface Σ_E is parametrized by the coordinates (q_1, p_1). It is well known that the Poincaré map φ_E is area-preserving and hence the eigenvalues of its linearized map (Jacobian matrix) occur in pairs λ and λ^{-1}.

Let us write down the system of equations for (1) and (4). For buckled beam case (1), it can be written as

$$\begin{aligned} \dot{q_1} &= p_1, \qquad \dot{p_1} = q_1 - {q_1}^3 - {j_2}^2 q_1 q_2^2 \\ \dot{q_2} &= p_2, \qquad \dot{p_2} = -{\omega_1}^2 q_2 - {j_2}^2 {q_1}^2 q_2. \end{aligned} \tag{5}$$

On the other hand, the Hénon-Heiles system can be written as

$$\dot{q_1} = p_1, \qquad \dot{p_1} = -q_1 - 2q_1q_2$$
$$\dot{q_2} = p_2, \qquad \dot{p_2} = -q_2 - {q_1}^2 + {q_2}^2. \tag{6}$$

If we set $q_1 = p_1 = 0$ for above systems, the first line of the equations are automaticallly satisfied and systems are reduced to the second line equations. In particular, the second line equations of (1) are written as linear equations

$$\dot{q_2} = p_2, \qquad \dot{p_2} = -{\omega_1}^2 q_2.$$

On the other hand, those of (4) are written as nonlinear equations

$$\dot{q_2} = p_2, \qquad \dot{p_2} = -q_2 - {q_2}^2.$$

In both cases, the solutions are periodic functions of the time t. If we write the energy relation $H = E(= \text{const.})$ in the form $p_2^2 = 2E - p_1^2 - 2V(q_1, q_2)$, the inequality $p_2^2 \geq 0$ implies

$$2E - {p_1}^2 + {q_1}^2 - {\omega_1}^2{q_2}^2 - \frac{1}{2}{q_1}^4 - {j_2}^2{q_1}^2{q_2}^2 \geq 0. \tag{7}$$

in the bucked beam case, and

$$2E - {p_2}^2 - {q_1}^2 - {q_2}^2 - 2c{q_1}^2 q_2 - \frac{2}{3}d{q_2}^3 \geq 0, \tag{8}$$

in the Hénon-Heiles case. From these inequalities, the points (q_1, p_1) on the cross section are restricted to the domains

$$\begin{cases} 2E - {p_1}^2 + {q_1}^2 - \frac{1}{2}{q_1}^4 \geq 0, & \text{(buckled beam case)} \\ 2E - {p_1}^2 - {q_1}^2 \geq 0. & \text{(Hénon-Heiles case)} \end{cases} \tag{9}$$

One can easily see that both of these domains are bounded.

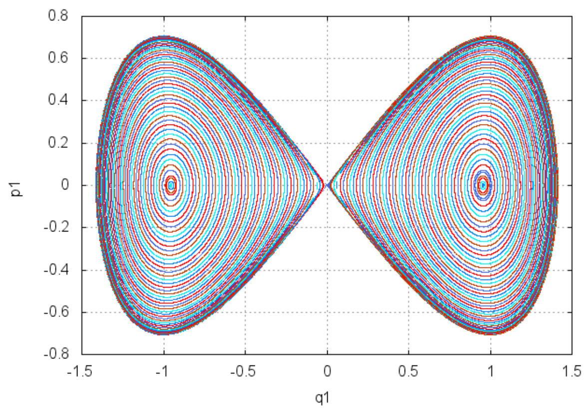

Figure 1: Orbits of φ_0 for the buckled beam system with $j_2 = 2$

We take initial points for the Poincaré map φ_E along q_1 axis ($p_1 = 0$) satisfying condition (9), in sequence form expressed by

$$\begin{cases} \sqrt{1+\sqrt{1+4E}} - 2i\dfrac{\sqrt{1+\sqrt{1+4E}}}{N}, i = 1, 2, ..., N & \text{(buckled beam case)} \\ \sqrt{2E} - 2i\dfrac{\sqrt{2E}}{N}, i = 1, 2, ..., N & \text{(Hénon-Heiles case),} \end{cases} \tag{10}$$

where N is the number of initial points. We use $N = 100$ for both cases. The structure of the orbits of the Poincaré map φ_0 of buckled beam system with $j_2 = 2$ is shown in Figure 1 above.

3 Stability of periodic points

We consider stability of periodic orbits satisfying $q_1(t) = p_1(t) = 0$ in buckled beams system. The orbit with energy E is given by

$$q_2(t) = \frac{\sqrt{E}}{\omega_1}\sin(\omega_1 t), \qquad p_2(t) = \sqrt{E}\omega_1\cos(\omega_1 t).$$

This solution has the period $2\pi/\omega_1$. The system of variational equations along these solutions (linearized system) is given by

$$\dot{\zeta} = JH_{zz}(q(t), p(t))\zeta, \qquad \zeta = (\xi_1, \xi_2, \eta_1, \eta_2) \in \mathbb{R}^4, \qquad J = \begin{pmatrix} O & I \\ -I & O \end{pmatrix},$$

where $H_{zz}(q(t), p(t))$ is the Hessian matrix of H at the point $(q(t), p(t))$ and I is the identity matrix of degree 2. Its normal variation part is the following

$$\dot{\xi_1} = \eta_1, \qquad \dot{\eta_1} = \left(1 - \frac{E}{2} + \frac{E}{2}\cos(2\omega_1 t)\right)\xi_1. \tag{11}$$

The Jacobian matrix $D\varphi_E$ of the Poincaré map at the origin $(q_1, p_1) = (0, 0)$ is similar to the matrix M whose first and second columns are the position at $t = \frac{2\pi}{\omega_1}$ of the orbit of (11) through $(\xi_1, \eta_1) = (1, 0)$ and $(0, 1)$ respectively. Then, we can analyse the type of the periodic point by calculating the trace of matrix M.

Let λ, λ^{-1} be the eigenvalues of M. Then, if λ is real, we have $|\operatorname{tr} M| \geq 2$ since $|\lambda + \lambda^{-1}| \geq 2$. On the other hand, if λ is imaginary, $\bar{\lambda}$ is also an eigenvalue of M. Therefore, we have $\bar{\lambda} = \lambda^{-1}$. This implies that $|\lambda| = 1$ and hence one can write $\lambda = \cos\theta + i\sin\theta$ for some real number θ, leading to

$$\operatorname{tr} M = 2\cos\theta.$$

Therefore, we claim that the periodic orbit is elliptic if $|\operatorname{tr} M| < 2$ and hyperbolic if $|\operatorname{tr} M| > 2$, and parabolic if $|\operatorname{tr} M| = 2$. Figure 2 below shows the orbit structure of the Poincaré map for buckled beam system when it has elliptic periodic points.

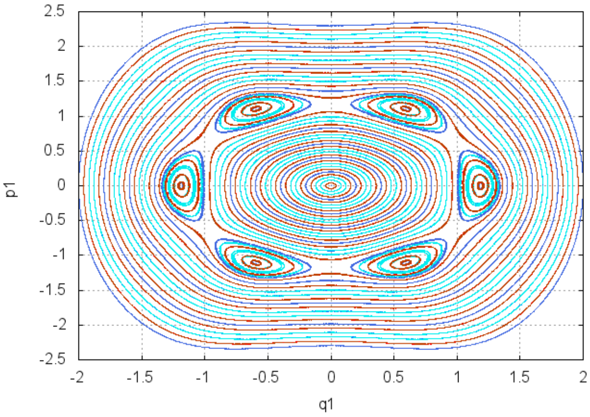

Figure 2: Orbits of $\varphi_{3.5}$ for buckled beam system with $j_2 = 2$

4 Lyapunov Exponent

Let $z(t, z_0) = (q(t), p(t))$ be an orbit through a point $z_0 \in \mathbb{R}^4$ of a given system and consider a nearby orbit $z(t, z_0')$ through the point z_0' close to z_0. Then $d_0(0) = z_0' - z_0$ denotes the vector from z_0 to z_0'. In the time step T for solving differential equations, this small vector $d_0(0)$ transforms into $d_0(T) := z(T, z_0') - z(T, z_0)$. Next we renormalize this vector $d_0(T)$ as new initial deviation vector $d_1(0)$ of the same length as $d_0(0)$, and then define $d_1(T)$ in the same way. This give the sequence of vectors $d_i(0)$ and $d_i(T)$ for $i = 0, 1, \ldots$ inductively. By using this method, the maximal Lyapunov exponent through the point z_0 is approximated by the following limit

$$\lambda = \lim_{n\to\infty} \frac{1}{nT} \sum_{i=0}^{n-1} \log \frac{|d_i(T)|}{|d_i(0)|}. \tag{12}$$

Note that, if $\lambda > 1$, we can presume that the orbit near initial point $z_0 = (q_0, p_0)$ is chaotic, because the deviation between nearby orbits and the given one becomes larger when the time nT is large enough. If the system is integrable, then for all choices of initial point, it must have Lyapunov exponent $\lambda \approx 0$.

5 Results and Discussions

The differential equations in this paper were solved using the Dorman-Prince integrator applying eight-order Runge-Kutta method [2]. Figure 3 below shows the behavior of the trace of M as a function of the energy for the buckled beam system with $j_2 = 2$.

We see that the stability types of periodic points change between hyperbolic and elliptic. For instance, when the energy $E = 1$, $\operatorname{tr} M = 4.99449 > 2$ ($\lambda = 4.78553, \lambda^{-1} = 0.208963$) and therefore the periodic point is hyperbolic. But it becomes elliptic when the energy $E = 2$ since $|\operatorname{tr} M| = 1.9224 < 2$ ($\lambda = 0.962201 + 0.272339I, \lambda^{-1} = 0.962201 - 0.272339I$) and still elliptic up to energy $E = 11$. The period of the change of stability type becomes larger as the energy increases.

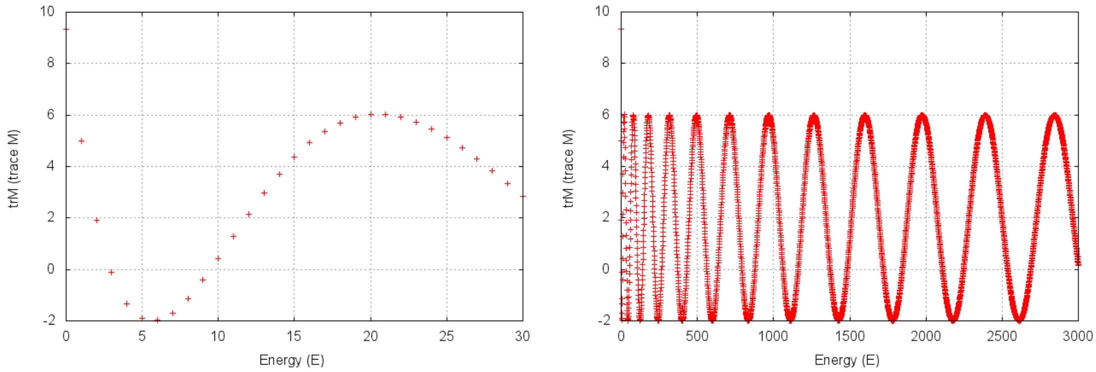

Figure 3: Trace of Jacobian matrix M with energy $0 \leq E \leq 30$ (left) and $0 \leq E \leq 3000$ (right)

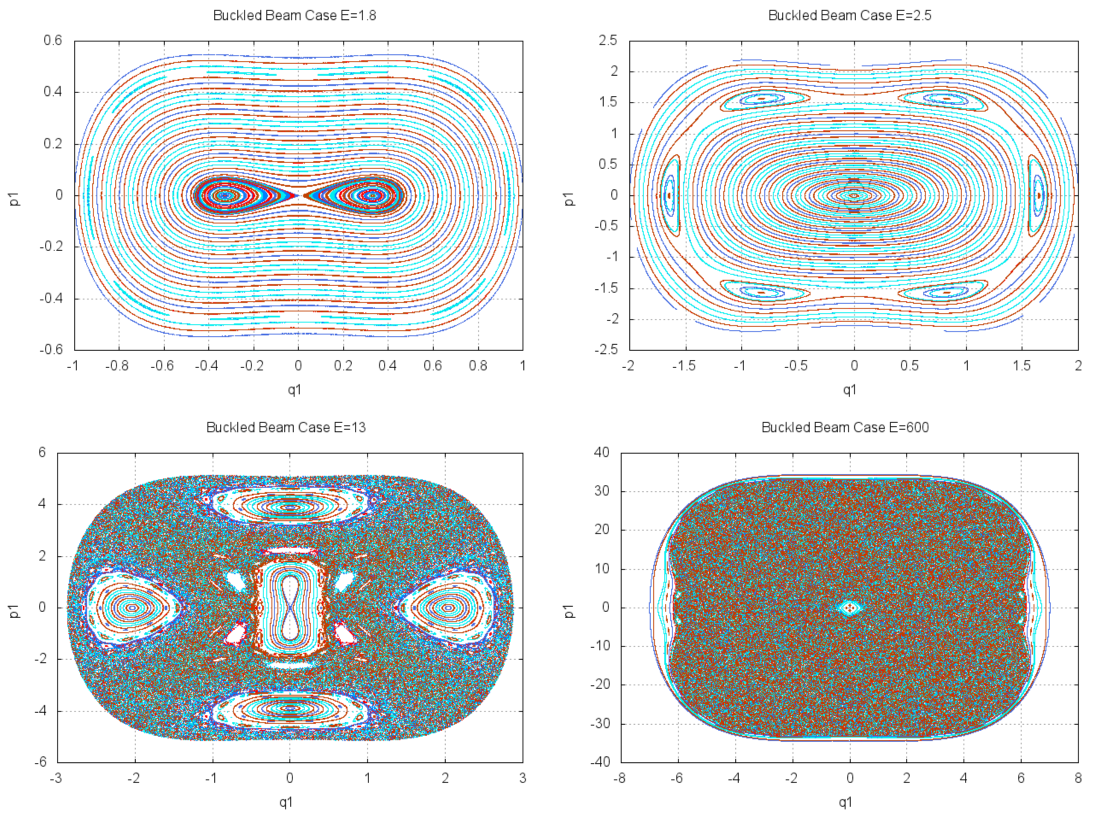

Figure 4: Orbits of $\varphi_{1.8}$, $\varphi_{2.5}$, φ_{13}, φ_{600} for buckled beam system with $j_2 = 2$

Figure 4 above shows orbits of the Poincaré map with several energy surfaces of buckled beam system with $j_2 = 2$. The phase space structure of a neighbourhood of the origin changes according to the values of the trace of M. For instance, when the energy $E = 1.8$, the periodic point $(0, 0)$ is hyperbolic and there is no invariant circle in a neighborhood of the origin. But, in contrast when the energy $E = 2.5$ there are invariant circles. A neighborhood of the origin containing invariant circles appear and disappear repeatedly as the energy increases, even though it is very small and surrounded by plenty of chaotic orbits.

The stability type of periodic points of buckled beam with $j_2 = 6$ behaves differently. It changes from hyperbolic to elliptic between energy $E = 31$ and $E = 40$, but then continues to be always elliptic as the energy increases. There are invariant circles in any neigborhood of the origin when $E > 40$. This fact is shown in Figure 5.

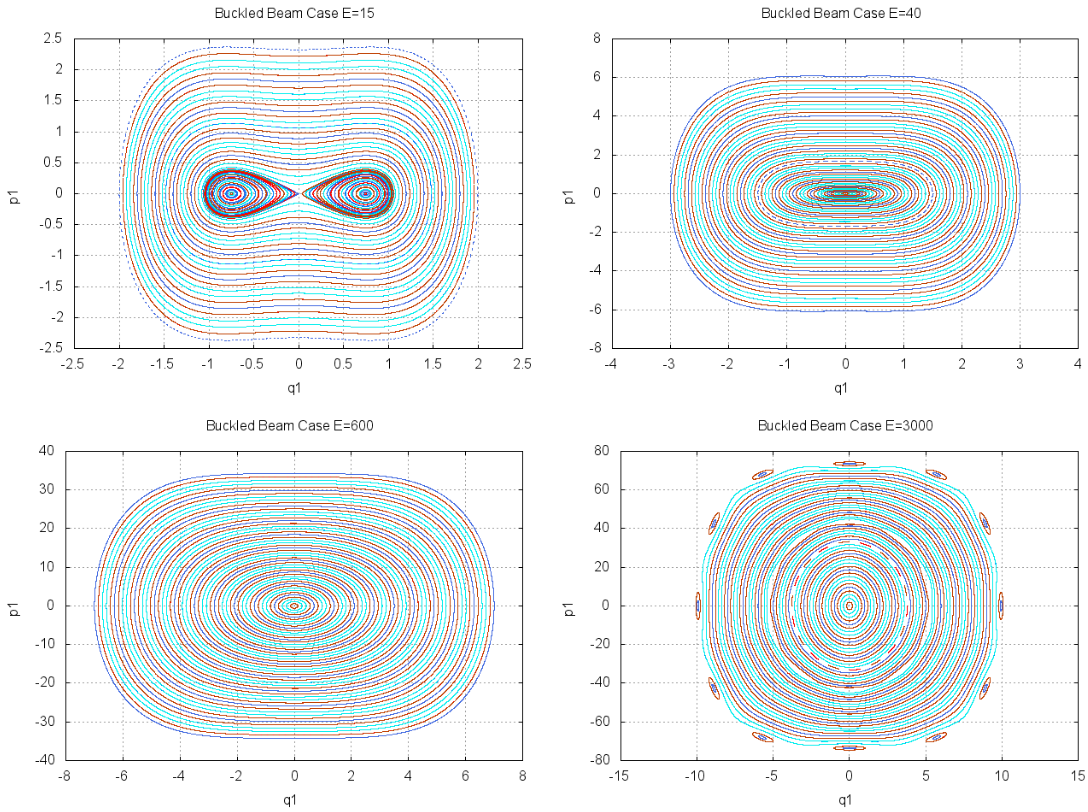

Figure 5: Orbits of φ_{15}, φ_{40}, φ_{600}, φ_{3000} for buckled beam system with $j_2 = 6$

The stability type of periodic point $(0, 0)$ of Hénon-Heiles system changes in a way different from the buckled beam case. When the energy $E = 0.08$, the origin is an elliptic fixed point which is surrounded by a lot of invariant circles and the same situation continues to exist as the energy increases. But, when the energy $E = 0.14$, those invariant circles disappear and the origin is a hyperbolic fixed point. The behavior of the trace function is shown in Figure 6 and the phase space structure of Hénon-Heiles system is shown in Figure 7.

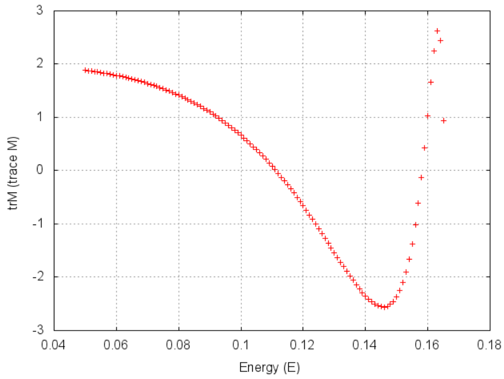

Figure 6: Trace of Jacobian matrix M for Hénon-Heiles system

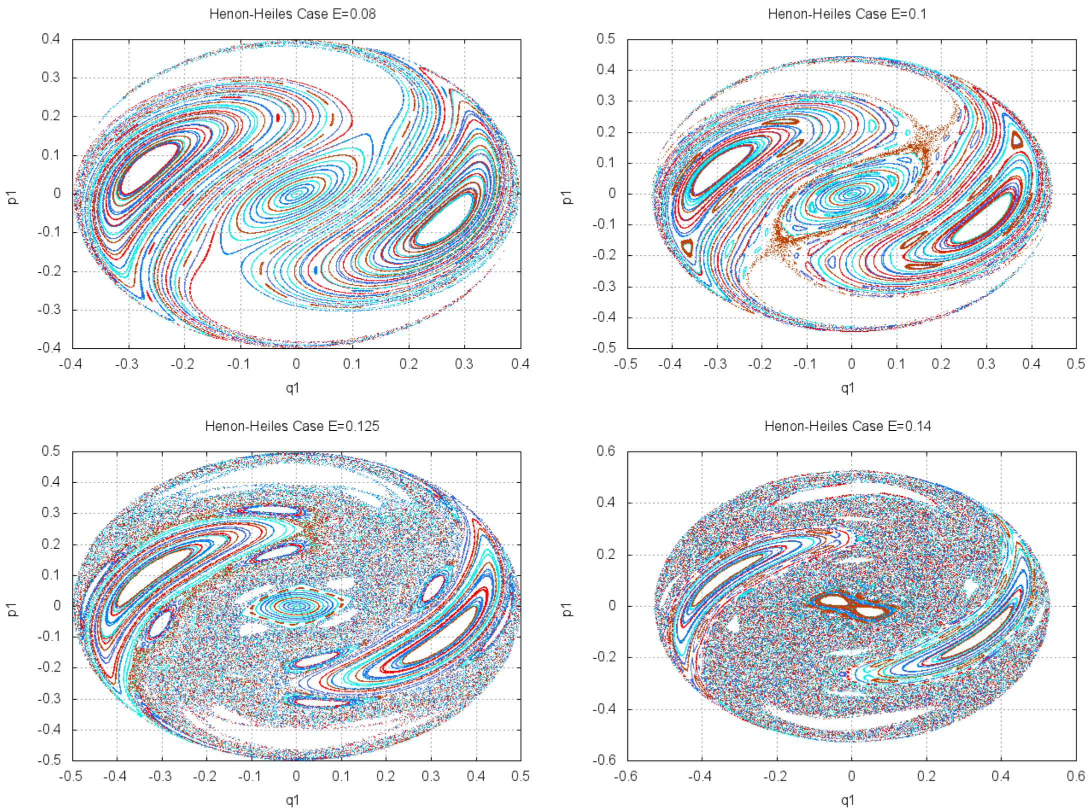

Figure 7: Orbits of $\varphi_{0.08}$, $\varphi_{0.1}$, $\varphi_{0.125}$, $\varphi_{0.14}$ for Hénon-Heiles system

Figure 8 below shows the graph of Lyapunov exponent as the energy increases of buckled beam system with $j_2 = 2$. When the periodic point is elliptic, the Lyapunov exponent $\lambda \approx 0$ for a point near the origin. But, when it is hyperbolic, the Lyapunov exponent $\lambda > 0$. This fact is shown in Figure 8 (left) with initial point $(q_1, p_1) = (0.05, 0)$. On the other hand, if the point is not periodic, then the Lyapunov exponent will grow up almost monotonically under the increase of energy. This fact is shown in Figure 8 (right) with initial point $(q_1, p_1) = (1, 0)$. This result corresponds to calculation of the Lyapunov exponent for Hénon-Heiles system obtained by Shevcenko [4]. Figure 9 shows different behavior of Lyapunov exponent of buckled beam system with $j_2 = 6$. This case seems to be integrable from the viewpoint of the fact that the Lyapunov exponent $\lambda \approx 0$ for almost all points under the increase of energy. These results confirm the (non)integrability analysis (Yagasaki [5]), showing the existence of transverse homoclinic orbits for $j_2 = 2$ yielding nonintegrability, while the case $j_2 = 6$ is not known yet if nonintegrable or not.

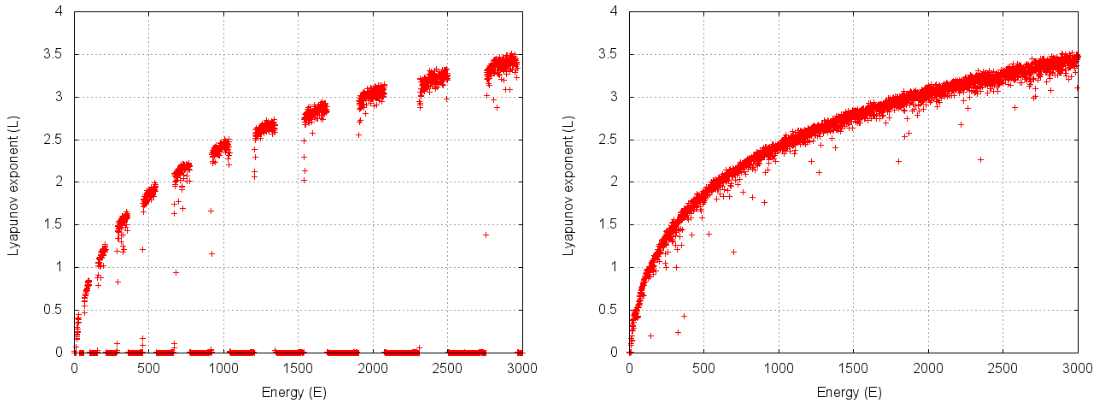

Figure 8: Lyapunov exponent of buckled beam with $j_2 = 2$ at $(q_1, p_1) = (0.05, 0)$ (left) and $(q_1, p_1) = (1, 0)$ (right)

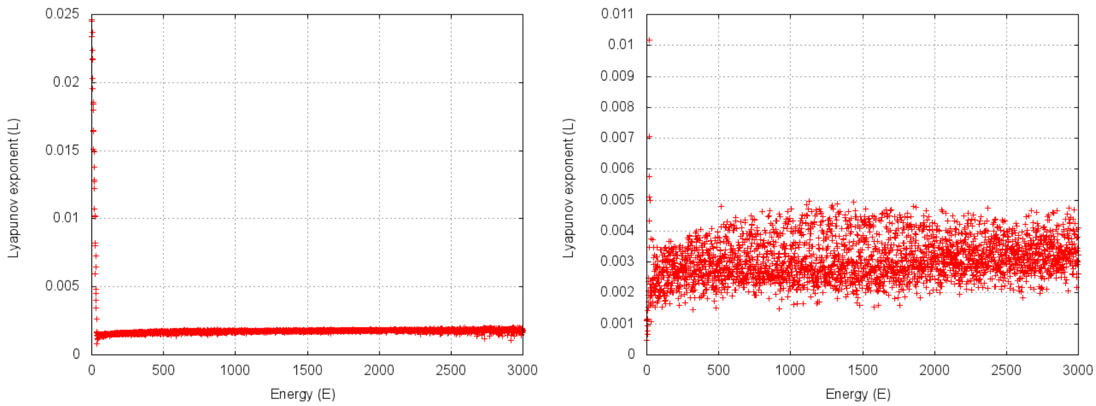

Figure 9: Lyapunov exponent of buckled beam with $j_2 = 6$ at $(q_1, p_1) = (0.05, 0)$ (left) and $(q_1, p_1) = (1, 0)$ (right)

6 Conclusion

We studied the stability change of periodic orbits and the phase space structure of buckled beam system with Hamiltonian of the form $H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(-q_1^2 + \omega_1^2 q_2^2) + \frac{1}{4}q_1^4 + \frac{j_2^2}{2}q_1^2 q_2^2$, particulary for case $j_2 = 2$ and 6. We analysed this system by using two-dimensional Poincaré maps. The periodic point of buckled beam system with $j_2 = 2$ changes repeatedly between hyperbolic and elliptic under the increase of energy, while the periodic point of buckled beam with $j_2 = 6$ is always elliptic when the energy $E > 40$. We compare these results to Hénon-Heiles system with Hamiltonian of the form $H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2) + q_1^2 q_2 - \frac{1}{3}q_2^3$. This system has no such repeated change of stability type of periodic point between energy $E = 0.08$ until $E = 0.170$.

The orbit structure of a neighbourhood of the origin changes according to the type of periodic point. If the periodic point is elliptic, then the orbit has invariant circles which is in contrast to the hyperbolic point where there is no invariant circle. The Lyapunov exponent of an initial point near the origin of buckled beam with $j_2 = 2$ also changes according to the type of periodic point. This suggests that the motion is irregular or random. On the other hand, the Lyapunov exponent at non periodic point grows almost monotonically. These results correspond to the Lyapunov exponent for Hénon-Heiles system obtained by [4]. Buckled beam with $j_2 = 6$ seems to be integrable since it has Lyapunov exponent $\lambda \approx 0$ for almost all points.

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